The factor objective under the mandate: spending a budget of tracking error on factor exposure in French's 49 industries

Carlo Hofer. November 2026. The second note of the project Optimal versus naive diversification. Code and notebooks: github.com/ochofer/optimal-vs-naive-diversification.

The first note of this project found that a long-only index-tracking mandate on French's 49 industries leaves small factor exposures that nobody chose. This note spends tracking error on factor exposure deliberately. Each month from 1979 to 2026 the optimiser maximises the portfolio's exposure to value, profitability, investment and momentum under the mandate, within a budget of 75, 100 or 150 basis points a year of forecast tracking error. At 100 basis points the combined exposure is 0.109 to 0.140, and the realised tracking error is 19 to 50 per cent above the budget. The targeted factors contributed +26.7 to +35.5 basis points a year, statistically significant at the 5 per cent level, and the six factors together +37 to +47, but the residual, i.e., the part of the active return the factors do not explain, was −31 to −37, so no path's active return is statistically significant at that level.

Terms

Benchmark: the cap-weighted portfolio of French's 49 industry portfolios. **Active weight**: an industry's portfolio weight minus its benchmark weight. **Active return**: the portfolio's return minus the benchmark's. **Tracking error**: the standard deviation of the monthly active return, times the square root of 12. It is forecast before the month and realised after it. **Basis point**: a hundredth of a percentage point. **Mandate**: the rules the portfolio obeys, set out in Section 2. **Beta**: the slope of an industry's return on a factor's return, estimated on the three years of daily returns before each month. **Active exposure**: the portfolio's beta to a factor minus the benchmark's, i.e., the sum over industries of the active weight times the industry's beta. **Targeted factors**: value, profitability, investment and momentum, each counted in the direction of its premium. **Factor objective**: the sum of the portfolio's active exposures to the four targeted factors, which the optimiser maximises. **Budget**: the limit on the forecast tracking error, 75, 100 or 150 basis points a year. **Path**: the month-by-month sequence of portfolios built with one covariance estimate at one budget. **Bias statistic**: the standard deviation of the realised active return divided by its forecast. It is 1 for an accurate forecast and above 1 when the forecast is too low. **Residual**: the part of the active return the six factors do not explain, i.e., the active return minus the six factor contributions. It comes from the industries' returns beyond what their factor exposures imply. **Factor share**: the share of the variance of the active return that the six factors explain. **Exposure per basis point**: a path's mean exposure to the targeted factors divided by its realised tracking error in basis points a year. **Gap**: the part of a month's factor objective that is lost when the same weights are valued with the betas estimated after the month.

1. Introduction

An enhanced index fund spends a budget of tracking error on purpose, taking factor exposures its benchmark does not hold while staying close to it (Robeco, 2024). The first note of this project (Hofer, 2026) measured what a long-only index-tracking mandate on French's 49 industries costs when nothing is spent on purpose: the portfolio closest to the benchmark realises 84 to 96 basis points a year of tracking error, and the mandate's tilt away from Coal, Oil and Utilities leaves small exposures to the common factors that nobody chose. The question this note asks is the one the holder of that mandate asks next: how much exposure to four factors a given budget of tracking error obtains, and what that exposure contributes to the return.

The factor literature documents the premia and their long-run averages (Fama and French, 2015; Carhart, 1997), and descriptions of enhanced indexing describe portfolios that take factor exposures within a tracking-error limit (Robeco, 2024). Neither shows, under a given mandate, how much exposure is obtained per basis point of tracking error when every industry must stay within 2 percentage points of its benchmark weight, how far the realised tracking error exceeds the budget the optimiser was held to, which factors the optimiser actually buys when all four are weighted equally, and what share of the resulting active return the factors explain when the assets are industries.

This note tests this on French's 49 industries. Each month from July 1979 to August 2026 the optimiser maximises the sum of the portfolio's active exposures to value, profitability, investment and momentum, under every rule of the mandate and a budget on the forecast tracking error of 75, 100 or 150 basis points a year, with the forecast from one of the first note's five covariance estimates. Seven expectations about the results were written before any path ran, and Table 5 lists them. An attribution then splits each path's active return into the contributions of six factors and a residual, the part the factors do not explain.

French's 49 industries are well suited to test this question, because the first note has already measured the mandate on the same months, so every figure here has a counterpart without the objective. At the same time, the industries' betas to

the four factors differ by about 0.5, so a bound of 2 percentage points per industry allows exposures of 0.03 to 0.08 per factor, large enough to measure and small enough for a realistic mandate.

Empirically, the budget is spent almost in full, but unevenly across the four factors. At 100 basis points the optimiser obtains 0.109 to 0.140 of exposure to the four factors together, 74 to 85 per cent of it in profitability and investment and almost none in momentum, and the realised tracking error is 118.6 to 137.0 basis points a year on the daily estimators and 150.1 on the monthly covariance, 19 to 50 per cent above the budget. The targeted factors contributed +26.7 to +35.5 basis points a year, statistically significant at the 5 per cent level on every estimator. The residual, i.e., the part of the active return the factors do not explain, was −31 to −37 against the six factors' +37 to +47, however, so no path's active return is statistically significant at that level, which at 100 basis points would take one of about 34 to 43 basis points a year.

2. Design

The data, the benchmark and the mandate are those of the first note. French's 49 value-weighted industry portfolios and his six factors come from his library, on the files built from the CRSP database of August 2026, and the benchmark is the cap-weighted portfolio of the 49 industries. The mandate holds the portfolio long-only, each industry within 2 percentage points of its benchmark weight, one-way turnover at most 2 per cent a month, Coal, Oil and Utilities at no more than half their benchmark weight, tobacco and weapons excluded, and the beta to the benchmark at one.

The objective replaces the first note's. Instead of the smallest forecast tracking error, the optimiser seeks the largest *factor objective*, the sum of the portfolio's active exposures to value, profitability, investment and momentum, each counted in the direction of its premium and weighted equally. An active exposure is the sum over industries of the active weight times the industry's beta to the factor, with each industry's betas estimated on its daily returns over the three years before the month. The market is left out because the mandate already holds the portfolio's beta at one, and size is not targeted. Every rule of the mandate still applies, and the forecast tracking error from the covariance in use must stay at or below the budget. In a month when even the portfolio closest to the benchmark exceeds the budget, the optimiser takes that portfolio, and the month counts as one in which the budget was not met.

Nineteen paths run over the 566 months, fifteen *main paths* and four more at 100 basis points. The *main paths* are the five covariance estimates of the first note at each of the three budgets: the sample covariance, Ledoit-Wolf shrinkage (Ledoit and Wolf, 2004), a six-factor model and a principal-component model on three years of daily returns, and the sample covariance of 120 monthly returns. The four more are the robust portfolio, whose budget holds under all five covariances at once, RiskMetrics, and the sample covariance targeting value alone and momentum alone. Each path is judged by its realised tracking error against the budget, its bias statistic, its active return with its standard error, its turnover and trading cost, and the exposures it holds.

The attribution splits each month's active return into six contributions, each the active exposure to a factor times the factor's return in the month, and a *residual*. Two measures describe how the budget is spent, the *exposure per basis point* and the *gap*. The *exposure per basis point* divides a path's mean exposure by its realised tracking error. The *gap* measures how much of the exposure chosen with the betas known before the month is lost when the same weights are valued with the betas estimated after it. Seven expectations were written into the repository before any path ran, and Table 5 states each with its verdict, met or not met. The notebooks mark the comparisons added after the results were known.

3. Results

3.1 The exposures obtained

Table 1 reports the nineteen paths and Table 2 the exposures they hold. At 100 basis points the budget binds, within one basis point, in 96.6 to 97.3 per cent of the months the mandate alone can meet on the four daily estimators (expectation E1, met). The mandate alone cannot meet the budget in 22 to 43 of the 566 months, nearly all in 1979 to 1989, when Oil held 15 per cent of the benchmark and the tilt alone produced more than 100 basis points of forecast tracking error. All four targeted exposures are positive on every estimator, and the mean exposure rises with the budget on every estimator

(E2, met). Against the exposures the mandate leaves behind on its own, between -0.015 and -0.004 on the four factors, the factor objective reverses the sign of all four and makes them three to five times as large.

The exposure is spread unevenly across the four factors. Per basis point of realised tracking error the five main paths at 100 basis points obtain 0.00092 to 0.00096 of exposure to the four together, but profitability and investment account for 74 to 85 per cent of it, value for most of the rest and momentum for almost none, 0.00001 to 0.00007. The four are weighted equally in the objective, and the optimiser buys most of the two whose industry betas it can raise at the least forecast tracking error within the bounds, and least of momentum, whose industry betas are the smallest and change fastest. An equal weighting in the objective therefore does not produce equal exposures. Targeting a single factor shows the trade-off: the value-only path obtains 0.00060 of value per basis point and loses 0.00030 of investment, and the momentum-only path obtains 0.00032 of momentum and loses enough value and investment that its exposure to the four together is 0.00002 per basis point, against 0.00096 for the four-factor path on the same estimate.

Table 1. The nineteen paths of the factor objective, 1979-07 to 2026-08 (566 months): tracking error, forecast bias, active return and trading

Covariance estimate	*Realised tracking error, bp a year*	*Mean forecast, bp a year*	*Realised over budget*	*Bias statistic*	*Active return, gross, bp a year (s.e.)*	*Trading cost at 50 bp, bp a year*	*Active return, net, bp a year*	*One-way turnover, % a month*	*Budget binds, months*	*Turnover cap binds, months*	*Budget not met, months*
Budget of 75 basis points a year											
Sample covariance, 3 years daily	100.2	79.2	1.34	1.22	-2.2 (14.6)	10.7	-12.8	1.78	449	330	116
Ledoit-Wolf shrinkage, 3 years daily	98.6	79.8	1.31	1.18	-3.1 (14.4)	10.3	-13.4	1.72	428	291	137
Six-factor model, 3 years daily	110.2	79.4	1.47	1.34	+3.7 (16.1)	9.0	-5.3	1.49	436	200	129
Principal components, 3 years daily	112.7	78.1	1.50	1.39	-3.9 (16.4)	10.3	-14.1	1.71	457	304	109
Sample covariance, 120 months	128.0	79.6	1.71	1.55	-0.9 (18.6)	11.8	-12.7	1.97	468	470	97
Budget of 100 basis points a year											
Sample covariance, 3 years daily	120.6	101.1	1.21	1.18	+8.9 (17.6)	11.5	-2.6	1.92	513	447	47
Ledoit-Wolf shrinkage, 3 years daily	118.6	101.2	1.19	1.16	+6.5 (17.3)	11.3	-4.8	1.88	509	418	50
Six-factor model, 3 years daily	133.6	100.7	1.34	1.32	+15.5 (19.5)	10.7	+4.8	1.78	509	378	50
Principal components, 3 years daily	137.0	100.8	1.37	1.35	+9.9 (19.9)	11.2	-1.3	1.87	527	427	33
Sample covariance, 120 months	150.1	101.8	1.50	1.45	+6.0 (21.9)	12.0	-6.0	2.00	517	518	47
Robust (worst case of the five)	115.6	102.7	1.16	1.08	+2.3 (16.8)	11.2	-8.8	1.86	493	410	73
RiskMetrics (exponentially weighted), 5 years daily	118.6	101.7	1.19	1.15	+7.5 (17.3)	11.8	-4.3	1.96	512	491	48
Sample covariance, 3 years daily, value alone	122.8	101.0	1.23	1.20	-14.5 (17.9)	11.2	-25.7	1.87	507	408	47
Sample covariance, 3 years daily, momentum alone	123.0	101.0	1.23	1.21	+7.8 (17.9)	11.6	-3.9	1.94	518	478	46
Budget of 150 basis points a year											
Sample covariance, 3 years daily	173.1	145.7	1.15	1.20	+33.9 (25.2)	11.9	+21.9	1.99	410	509	7
Ledoit-Wolf shrinkage, 3 years daily	171.3	146.7	1.14	1.18	+34.0 (24.9)	11.9	+22.1	1.98	430	510	8
Six-factor model, 3 years daily	187.7	143.1	1.25	1.31	+33.9 (27.3)	11.5	+22.3	1.92	356	469	9
Principal components, 3 years daily	185.0	144.3	1.23	1.29	+27.3 (26.9)	11.8	+15.5	1.96	389	499	10
Sample covariance, 120 months	191.7	148.8	1.28	1.28	+20.5 (27.9)	12.1	+8.4	2.02	455	537	7

Realised tracking error is the standard deviation of the monthly active return times the square root of 12, and the forecast is the mean of the optimiser's monthly forecasts. Realised over budget divides the realised figure by the budget. The bias statistic is the standard deviation of the monthly active return divided by the same month's forecast, 1 for an accurate forecast. The standard error of an active return is its realised tracking error over the square root of 47 years. Trading cost is charged at 50 basis points per unit of one-way turnover, and the net return is the gross less the mean cost. The budget binds in a month when the forecast is within one basis point of it, and it is not met in a month when the portfolio closest to the benchmark already exceeds it or the turnover cap stops the path short of it. The benchmark itself turns over 0.39 per cent a month.

Table 2. Mean active exposures to the targeted factors, in beta units, and the exposure per basis point of realised tracking error

Covariance estimate	*Value (HML)*	*Profitability (RMW)*	*Investment (CMA)*	*Momentum (Mom)*	*The four together*	*Per bp, the four*	*Of which value*	*Profitability*	*Investment*	*Momentum*
Budget of 75 basis points a year										
Sample covariance, 3 years daily	+0.006	+0.025	+0.022	+0.004	+0.057	0.00057	0.00006	0.00025	0.00022	0.00004
Ledoit-Wolf shrinkage, 3 years daily	+0.003	+0.021	+0.020	+0.003	+0.047	0.00048	0.00003	0.00021	0.00021	0.00003
Six-factor model, 3 years daily	+0.010	+0.028	+0.022	+0.004	+0.064	0.00058	0.00009	0.00025	0.00020	0.00004
Principal components, 3 years daily	+0.009	+0.028	+0.022	+0.005	+0.064	0.00057	0.00008	0.00025	0.00020	0.00004
Sample covariance, 120 months	+0.005	+0.035	+0.041	-0.001	+0.081	0.00063	0.00004	0.00027	0.00032	0.00000
Budget of 100 basis points a year										
Sample covariance, 3 years daily	+0.021	+0.048	+0.039	+0.008	+0.116	0.00096	0.00017	0.00040	0.00033	0.00006
Ledoit-Wolf shrinkage, 3 years daily	+0.019	+0.045	+0.038	+0.007	+0.109	0.00092	0.00016	0.00038	0.00032	0.00006
Six-factor model, 3 years daily	+0.023	+0.051	+0.040	+0.009	+0.123	0.00092	0.00017	0.00038	0.00030	0.00007
Principal components, 3 years daily	+0.024	+0.053	+0.043	+0.009	+0.128	0.00094	0.00018	0.00038	0.00031	0.00006
Sample covariance, 120 months	+0.018	+0.056	+0.063	+0.002	+0.140	0.00093	0.00012	0.00038	0.00042	0.00001
Robust (worst case of the five)	+0.014	+0.040	+0.038	+0.004	+0.095	0.00082	0.00012	0.00034	0.00032	0.00004
RiskMetrics (exponentially weighted), 5 years daily	+0.022	+0.047	+0.041	+0.007	+0.116	0.00098	0.00018	0.00039	0.00034	0.00006
Sample covariance, 3 years daily, value alone	+0.074	-0.009	-0.037	-0.005	+0.023	0.00019	0.00060	-0.00007	-0.00030	-0.00004
Sample covariance, 3 years daily, momentum alone	-0.011	-0.007	-0.019	+0.039	+0.002	0.00002	-0.00009	-0.00006	-0.00015	0.00032
Budget of 150 basis points a year										
Sample covariance, 3 years daily	+0.035	+0.076	+0.064	+0.011	+0.186	0.00107	0.00020	0.00044	0.00037	0.00006
Ledoit-Wolf shrinkage, 3 years daily	+0.035	+0.075	+0.063	+0.011	+0.184	0.00107	0.00020	0.00044	0.00037	0.00006
Six-factor model, 3 years daily	+0.037	+0.079	+0.064	+0.011	+0.191	0.00102	0.00020	0.00042	0.00034	0.00006
Principal components, 3 years daily	+0.037	+0.078	+0.063	+0.012	+0.191	0.00103	0.00020	0.00042	0.00034	0.00007
Sample covariance, 120 months	+0.035	+0.082	+0.081	+0.004	+0.203	0.00106	0.00018	0.00043	0.00042	0.00002

An active exposure is the sum over industries of the active weight times the industry's beta to the factor, from a regression of its daily excess returns on the six factors over the three years before the month. The table shows its mean over the 566 months, each factor counted in the direction of its premium, and the four together are their sum. The exposure per basis point divides the mean exposure by the path's realised tracking error in basis points a year.

3.2 The tracking error realised

The realised tracking error exceeds the budget on every path. At 100 basis points the ratio of the realised tracking error to the budget is 1.19 to 1.37 on the daily estimators and 1.50 on the monthly covariance. At 150 it is 1.14 to 1.25 and 1.28, and at 75 it is 1.31 to 1.50 and 1.71. Expectation E3 asked for 1.2 to 1.6 on the daily estimators at every budget and is not met, because three of the twelve daily ratios fall below that range, so the forecast underestimated the tracking error by less than expected. The direction is the first note's, since an optimiser picks the weights for which the forecast

is lowest relative to the truth (Michaud, 1989). The size is smaller, because a portfolio held at a budget is less of an in-sample minimum than the portfolio with the smallest forecast: the bias statistics at 100 basis points are 1.16 to 1.35 on the daily estimators, against the first note's 1.22 to 1.39. The robust portfolio is again the best calibrated, at 1.08, and the monthly covariance the worst at 100 basis points, at 1.45.

The choice of estimator matters for the realised tracking error and little for the exposure obtained. At 100 basis points the realised tracking errors of the four daily estimators span 18.4 basis points on a level of about 130, in the first note's order, the sample covariance and shrinkage lowest and the two structured models highest, while their exposure per basis point lies within 4.5 per cent of its mean (E7, met). The turnover cap binds in 378 to 518 of the 566 months, the portfolios turn over 1.78 to 2.00 per cent a month one way against the benchmark's 0.39, and at 50 basis points per unit of turnover the trading costs 10.7 to 12.0 basis points a year.

3.3 The active return, and where it came from

Table 3 reports the attribution. At 100 basis points the gross active return is +6.0 to +15.5 basis points a year on standard errors of 17.3 to 21.9. No path's active return reaches 1.4 standard errors, so none is statistically significant at the 5 per cent level, and net of trading costs the return at 100 basis points is -6.0 to +4.8 (E5: the gross return is positive on all five estimates, as expected, and trading costs more than the 10 basis points expected). The four targeted factors contributed +26.7 to +35.5 basis points a year at 100 and +49.9 to +58.9 at 150, statistically significant at the 5 per cent level on all ten of those paths, and +5.7 to +16.6 at 75, which is not statistically significant on any of the five, where it would need about 16 to 26 basis points a year to be. These contributions are the exposures times the factors' realised returns over these 47 years, and the study estimates no premium.

The attribution's main finding is that the residual offsets most of the six factors' contribution. The residual is negative on every one of the nineteen paths, -10 to -47 basis points a year, and at 100 basis points it offsets 31 to 37 of the +37 to +47 that the six factors contribute together. The two parts move against each other because the same active weights carry both: in months when the targeted factors returned more, the industries held to obtain the exposures returned less than their factor exposures imply. As a result the six factors explain 14 to 28 per cent of the variance of the active return at 100 and 150 basis points, where at least 30 per cent was expected (E4, not met in this clause), close to the 14 to 25 per cent of the mandate alone. Table 4 shows how: the factors' own variance is 34 to 52 per cent of the active return's, the residual's 72 to 86 per cent, and their covariance takes 18 to 26 per cent off the sum. By decade, at 100 basis points, the targeted factors contributed most in the 1990s, +70 to +93 basis points a year, and the residual was most negative in the same decade, -104 to -143, so the decade's active return lies between -17 and +14.

The two factors outside the objective are not at zero either. The beta constraint holds the portfolio's beta to the benchmark, the cap-weighted 49 industries, at one, and the industries the optimiser buys move more with French's market factor than those it sells. The active exposure to the market factor is therefore 0.012 to 0.015 at 100 basis points on the daily estimators, and it contributed +9.8 to +11.7 basis points a year. Size contributed +1.2 to +2.8.

Table 3. Where the active return came from: the six factor contributions, the targeted total, the residual and the active return, in basis points a year

Covariance estimate	*Mkt-RF*	*SMB*	*HML*	*RMW*	*CMA*	*Mom*	*Targeted total (s.e.)*	*Factor total (s.e.)*	*Residual (s.e.)*	*Active return (s.e.)*
Budget of 75 basis points a year										
Sample covariance, 3 years daily	+5.5	+1.4	-4.0	+3.2	+8.4	+0.9	+8.5 (8.4)	+15.5 (8.3)	-17.7 (13.1)	-2.2 (14.6)
Ledoit-Wolf shrinkage, 3 years daily	+5.1	+2.0	-3.5	+1.5	+7.8	-0.1	+5.7 (8.3)	+12.8 (8.2)	-15.8 (12.8)	-3.1 (14.4)
Six-factor model, 3 years daily	+7.0	+1.5	-4.4	+5.1	+9.4	+1.5	+11.6 (8.8)	+20.1 (8.6)	-16.4 (14.9)	+3.7 (16.1)
Principal components, 3 years daily	+6.0	+0.4	-5.3	+4.2	+11.4	+1.1	+11.4 (9.6)	+17.8 (9.4)	-21.6 (14.7)	-3.9 (16.4)
Sample covariance, 120 months	-2.3	+2.2	+2.6	+6.3	+13.6	-5.9	+16.6 (13.2)	+16.4 (12.2)	-17.3 (16.6)	-0.9 (18.6)
Budget of 100 basis points a year										
Sample covariance, 3 years daily	+10.2	+2.3	-0.9	+13.5	+12.6	+3.9	+29.1 (11.1)	+41.6 (10.8)	-32.7 (15.9)	+8.9 (17.6)
Ledoit-Wolf shrinkage, 3 years daily	+9.8	+2.8	-1.0	+12.3	+12.2	+3.2	+26.7 (10.9)	+39.2 (10.7)	-32.8 (15.6)	+6.5 (17.3)
Six-factor model, 3 years daily	+11.7	+2.3	-1.5	+16.0	+13.6	+5.2	+33.3 (11.8)	+47.3 (11.4)	-31.8 (18.0)	+15.5 (19.5)
Principal components, 3 years daily	+10.6	+1.2	-2.9	+16.9	+16.6	+4.9	+35.5 (12.4)	+47.3 (12.1)	-37.4 (18.0)	+9.9 (19.9)
Sample covariance, 120 months	-0.5	+2.3	+6.2	+15.6	+18.5	-5.4	+34.8 (16.7)	+36.7 (15.3)	-30.7 (19.1)	+6.0 (21.9)
Robust (worst case of the five)	+8.5	+0.5	+1.0	+8.8	+12.1	-0.2	+21.7 (11.1)	+30.7 (10.7)	-28.4 (15.1)	+2.3 (16.8)
RiskMetrics (exponentially weighted), 5 years daily	+8.7	+1.1	-0.2	+14.1	+11.3	+3.0	+28.2 (10.8)	+38.1 (10.5)	-30.6 (15.9)	+7.5 (17.3)
Sample covariance, 3 years daily, value alone	+9.3	+1.1	+8.7	-5.1	-3.9	-8.0	-8.3 (11.3)	+2.0 (12.3)	-16.5 (15.2)	-14.5 (17.9)
Sample covariance, 3 years daily, momentum alone	-4.8	+3.4	-11.7	-3.8	+2.8	+32.2	+19.5 (10.1)	+18.1 (10.7)	-10.3 (16.0)	+7.8 (17.9)
Budget of 150 basis points a year										
Sample covariance, 3 years daily	+13.8	+8.1	+2.7	+23.9	+20.0	+4.2	+50.8 (15.9)	+72.7 (16.1)	-38.9 (22.7)	+33.9 (25.2)
Ledoit-Wolf shrinkage, 3 years daily	+14.3	+8.3	+2.7	+23.5	+19.6	+4.1	+49.9 (15.8)	+72.5 (15.9)	-38.5 (22.5)	+34.0 (24.9)
Six-factor model, 3 years daily	+15.9	+8.7	+2.2	+26.3	+22.0	+5.0	+55.5 (16.4)	+80.1 (16.7)	-46.3 (24.5)	+33.9 (27.3)
Principal components, 3 years daily	+14.8	+6.7	-0.8	+25.3	+23.0	+5.2	+52.8 (16.6)	+74.3 (16.9)	-47.0 (24.2)	+27.3 (26.9)
Sample covariance, 120 months	-3.3	+7.5	+13.6	+26.1	+23.4	-4.2	+58.9 (21.5)	+63.1 (20.2)	-42.6 (23.8)	+20.5 (27.9)

A contribution is the active exposure to the factor times the factor's return in the month, averaged and stated per year. The targeted total is the sum of the four targeted factors' contributions, the factor total the sum of all six, and the residual the active return minus the six, the part the factors do not explain. Contributions plus residual equal the active return in every month. Standard errors are the standard deviation of the monthly series over the square root of the number of months, times 12. Mkt-RF is the market factor, SMB size, HML value, RMW profitability, CMA investment and Mom momentum (Fama and French, 2015; Carhart, 1997).

3.4 The exposure lost when the betas move

Table 4 also reports the gap. At 100 basis points, valuing the weights with the betas estimated after the month reduces the exposure chosen before it by about 4 per cent, 0.039 to 0.044 per unit of exposure, and by more at 75 basis points, 0.054 to 0.076, where the exposure is smaller and the betas move as much. The largest monthly gaps fall in October 1990, July 2005, September 1981 and October 1987, months in which a large shock entered or left the three-year window the betas are estimated on. Momentum alone has the higher turnover, 1.94 per cent a month against value alone's 1.87, and its turnover cap binds more often, in 478 months against 408, but its gap in beta units is smaller, 0.00185 against 0.00235 (E6, not met in this clause). Per unit of exposure held the order is the expected one, 0.047 against 0.032, because the momentum-only path holds about half as much exposure.

Table 4. The factor share and the three shares of the active return's variance behind it, and the gap per unit of exposure bought

Covariance estimate	*Factor share*	*Factor total's own variance*	*Residual's own variance*	*Twice their covariance*	*Mean absolute gap, beta units*	*Mean factor objective*	*Gap per unit*	*Largest gap, month*
Budget of 75 basis points a year								
Sample covariance, 3 years daily	0.20	0.33	0.80	-0.13	0.00371	0.0571	0.065	1981-09
Ledoit-Wolf shrinkage, 3 years daily	0.20	0.33	0.80	-0.13	0.00359	0.0471	0.076	1981-09
Six-factor model, 3 years daily	0.14	0.29	0.86	-0.15	0.00427	0.0641	0.067	1990-10
Principal components, 3 years daily	0.20	0.33	0.80	-0.13	0.00411	0.0642	0.064	1990-10
Sample covariance, 120 months	0.21	0.43	0.79	-0.22	0.00441	0.0810	0.054	2005-07
Budget of 100 basis points a year								
Sample covariance, 3 years daily	0.18	0.38	0.82	-0.20	0.00468	0.1158	0.040	1990-10
Ledoit-Wolf shrinkage, 3 years daily	0.18	0.38	0.82	-0.20	0.00455	0.1089	0.042	2005-07
Six-factor model, 3 years daily	0.14	0.34	0.86	-0.20	0.00534	0.1227	0.044	1990-10
Principal components, 3 years daily	0.19	0.37	0.81	-0.18	0.00521	0.1283	0.041	1990-10
Sample covariance, 120 months	0.23	0.49	0.77	-0.26	0.00544	0.1401	0.039	2005-07
Robust (worst case of the five)	0.19	0.41	0.81	-0.21	0.00448	0.0953	0.047	2005-07
RiskMetrics (exponentially weighted), 5 years daily	0.15	0.37	0.85	-0.22	0.00480	0.1162	0.041	2005-07
Sample covariance, 3 years daily, value alone	0.27	0.47	0.73	-0.20	0.00235	0.0741	0.032	1990-10
Sample covariance, 3 years daily, momentum alone	0.20	0.36	0.80	-0.16	0.00185	0.0391	0.047	1987-10
Budget of 150 basis points a year								
Sample covariance, 3 years daily	0.19	0.41	0.81	-0.22	0.00645	0.1856	0.035	1990-10
Ledoit-Wolf shrinkage, 3 years daily	0.19	0.41	0.81	-0.22	0.00636	0.1838	0.035	1990-10
Six-factor model, 3 years daily	0.20	0.37	0.80	-0.18	0.00710	0.1906	0.037	1990-10
Principal components, 3 years daily	0.19	0.40	0.81	-0.20	0.00684	0.1908	0.036	1990-10
Sample covariance, 120 months	0.28	0.52	0.72	-0.25	0.00694	0.2026	0.034	1990-10

The factor share is one minus the variance of the residual over the variance of the active return. The three shares are the factor total's own variance, the residual's own variance and twice their covariance, each over the variance of the active return, and they sum to one. The gap of a month is the factor objective chosen with the betas estimated before the month minus the value the same active weights carry with the betas estimated after it. The mean absolute gap is over the 565 months that have a following month, and the gap per unit divides it by the path's mean factor objective.

Table 5. The seven expectations, written before any path ran, and what the run found

Expectation	*Written before the run*	*Found, in the words of the notebook that judged it*	*Verdict*
E1	At 100 basis points the budget binds, within one basis point, in at least 95 per cent of the months the mandate alone can meet, for each of the four daily estimators.	binding share over the months the mandate alone can meet, four daily estimators at 100 bp: 0.9698, 0.9732, 0.9658, 0.9688	MET
E2	At 100 basis points every targeted exposure is positive on the five estimators, the average of the four is at least 0.024 for the sample covariance, and the average rises with the budget on every estimator.	every targeted exposure positive on 5 of 5. average of the four for the sample covariance 0.0289. rises with the budget for 5 of 5	MET
E3	Realised tracking error exceeds the budget by 20 to 60 per cent on the daily estimators, and by more on the monthly covariance, at every budget.	daily ratios inside the range in 9 of 12. monthly above the largest daily at 3 of 3 budgets	NOT MET
E4	The four targeted contributions sum to a positive amount a year on every main path, and at 100 and 150 basis points the six factors explain at least 30 per cent of the active return's variance on every main path.	targeted total positive on 15 of 15 main paths. factor share at least 0.30 on 0 of 10 main paths at 100 and 150 bp (0.1439 to 0.2751)	NOT MET
E5	At 100 basis points the gross active return is positive on at least four of the five estimators, and the net return at 50 basis points per unit of turnover lies within 10 basis points a year of the gross.	gross positive for 5 of 5. net within 10 bp for 0 of 5	NOT MET
E6	Momentum alone has a higher one-way turnover and a larger gap than value alone, and the turnover cap binds in more months on the momentum path.	turnover higher, cap binds more, gap not larger on the momentum path	NOT MET
E7	At 100 basis points the realised tracking errors of the four daily estimators lie within 20 basis points of each other, the exposure bought per basis point differs across them by less than 30 per cent of its mean, and the monthly covariance is the worst calibrated of the five.	spread 18.41 bp. exposure per bp differs by 0.0447 of the mean. worst calibrated: the sample covariance, 120 months (1.45)	MET

The expectations are in `src/bp/constants.py`. E1, E2, E3, E5, E6 and E7 are judged in notebook 15 and E4 in notebook 16, and the verdicts are in `factor_objective_verdicts_202608.csv` and `factor_objective_attribution_verdict_202608.csv`.

4. Discussion

The natural comparison is with the first note's mandate on the same months. There the tracking error, 84 to 96 basis points a year, came from rules not chosen for return, and the exposures the tilt left behind contributed -9 to -13 basis points a year while the residual contributed +12 to +20. Here the tracking error is spent deliberately, and the estimates indicate that the optimiser obtains the exposures the arithmetic implies: a budget of 100 basis points yields 0.109 to 0.140 of exposure, and the four targeted factors contribute +26.7 to +35.5 basis points a year. The comparison also shows what the factor objective does not change. The residual remains the larger part of the variance of the active return and moves against the factor part, so the active return, +6 to +16 basis points a year gross at 100 on standard errors of 17 to 22, is not statistically significant at the 5 per cent level on any path, as the mandate's was not.

For the holder of such a mandate, the estimates therefore suggest three things. First, when the budget is spent on equally weighted factors, the optimiser obtains mostly the factors whose industry betas it can raise at little tracking error, here profitability and investment, so an equal weighting in the objective does not mean equal exposures in the portfolio. Second, the realised tracking error of a portfolio held at a budget by its own forecast exceeds the budget, by 14 to 50 per cent on the daily estimators and 28 to 71 per cent on the monthly covariance, so a budget written on the forecast limits a smaller tracking error than the one realised. Third, the targeted factors contributed what the sample's factor returns imply, but the active return a client receives is that contribution net of the residual, and over these 47 years it is not statistically significant at the 5 per cent level on any path.

At the same time, the design has limits. First, the assets are industry portfolios, so each basis point of tracking error yields about 0.001 of exposure, and at 150 basis points the mandate's bounds, and not the budget, limit the exposure in 104 to 201 of the 566 months. Whether a stock-level mandate obtains more exposure per basis point is a question these data cannot answer. Second, the betas come from three years of daily returns, and both the exposure obtained and the gap depend on that window. Third, the factor share depends on its definition: as one minus the residual's share, the definition of the first note, it is 0.14 to 0.28, while the factors' own variance alone is 0.34 to 0.52 of the active return's, and Table 4 reports both. Finally, the targeted factors' contribution is their realised returns over these 566 months times the exposures, so whether its sign would hold over another sample is a question these data cannot answer.

Against this evidence, the claim is a narrow one. On French's 49 industries, under the first note's mandate, a budget of forecast tracking error spent on four factors obtained positive exposure to each of them, mostly in profitability and investment, at a realised tracking error 14 to 71 per cent above the budget. The targeted factors' contributions were positive on every main path and statistically significant at the 5 per cent level at 100 and 150 basis points, the residual offset most of the six factors' contribution, and no path's active return is statistically significant at that level.

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